



# Decompositions of complete graphs into blown-up cycles $C_m[2]$

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## ABSTRACT

Let  $C_m[\overline{K}_2]$  stand for a cycle  $C_m$  in which every vertex is replaced by two isolated vertices and every edge by  $K_{2,2}$ . We prove that the complete graph  $K_{8mk+1}$  can be decomposed into graphs isomorphic to  $C_m[\overline{K}_2]$  for any  $m \geq 3$ ,  $k > 0$ . Decompositions of complete graphs into certain collections of even cycles are obtained as a corollary. Also some special cases of Alspach Conjecture are solved in this article. All proofs are constructive and use both graph theory and design theory techniques.

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## 1. Introduction

In this paper we consider only finite undirected graphs without loops and multiple edges. By a  $G$ -decomposition of the complete graph  $K_n$  (or a decomposition of  $K_n$  into  $G$ ) we understand a collection of subgraphs  $G_i$ ,  $i = 1, 2, \dots, s$  of  $K_n$  such that  $G_i \cong G$  for  $i = 1, 2, \dots, s$ ;  $E(G_1) \cup E(G_2) \cup \dots \cup E(G_s) = E(K_n)$ , and  $E(G_i) \cap E(G_j) = \emptyset$  for  $i \neq j$ , where  $E(H)$  denotes the edge set of a graph  $H$ . In the language of design theory, a  $G$ -decomposition of  $K_n$  is called  $(K_n, G)$ -design or just  $G$ -design.

### 1.1. Summary of known results

Decompositions of complete graphs  $K_n$  into cycles have been extensively studied by many authors. Clearly, for  $n$  even, no cycle decomposition (or any 2-regular decomposition) is possible. A natural extension is to look for a cycle decomposition of  $K_n - I$ , where  $I$  is a 1-factor of  $K_n$ . For decompositions into uniform cycles a complete characterization is known. The necessary conditions for a cycle  $C_m$  to decompose a complete graph  $K_n$  (or  $K_n - I$ , respectively) is that  $m$  divides the number of edges of  $K_n$  and  $m \leq n$ . For both cases of  $n$  odd and even it was shown by Alspach and Gavlas [1] and Šajna [12] that the necessary conditions are also sufficient.

**Theorem 1.1.** *Let  $n, m$  be integers with  $n \geq m \geq 3$ . Let  $K$  stand for  $K_n$  if  $n$  is odd and for  $K_n - I$  if  $n$  is even. Then  $K$  can be decomposed into  $C_m$  whenever  $m$  divides the number of edges in  $K$ .*

Obviously, for bipartite graphs no decomposition into odd cycles is possible. The even-cycle decomposition of  $K_{m,n}$  was completely solved by Bermond, Huang, and Sotteau [2,13].

**Theorem 1.2.**  *$K_{n_1, n_2}$  can be decomposed into  $C_m$  if and only if  $m, n_1, n_2$  are all even,  $m$  divides  $n_1 n_2$ , and both  $n_1, n_2 \geq \frac{m}{2}$ .*

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By  $G[H]$  we denote the composition (also called wreath product or lexicographic product) of graphs  $G$  and  $H$  which is obtained by replacing every vertex of  $G$  by a copy of  $H$  and every edge of  $G$  by the complete bipartite graph  $K_{|H|,|H|}$ . We say that  $G[H]$  arose from  $G$  by *blowing up* by  $H$  and recall that  $\overline{K_m}$  stands for the complement of  $K_m$ , i.e.  $m$  independent vertices. We often use the following corollary of Theorem 1.2.

**Corollary 1.3.** *A cycle  $C_m$  decomposes  $C_k[\overline{K_m}]$  for every even  $m \geq 3$ .*

**Proof.** We can decompose  $C_k[\overline{K_m}]$  into  $k$  copies of  $K_{m,m}$  and by Theorem 1.2 every  $K_{m,m}$  is decomposable into  $C_m$ .  $\square$

A collection of results on graph products was given in [9] by A. Muthusamy and P. Paulraja. They have shown the following.

**Theorem 1.4.** *If  $m$  and  $k$  are integers,  $k \geq 3$ , then  $C_k$  decomposes  $C_k[\overline{K_m}]$ .*

In fact the results given in [9] are more general. The authors have shown the existence (or proved nonexistence) of factorizations of  $C_k[\overline{K_m}]$  into collections of cycles of equal length  $p$ , where  $k \mid p$ . The following corollary will be particularly useful.

**Corollary 1.5.** *Let  $m, k$  be integers,  $k \geq 3$ . If  $C_k$  decomposes  $G$ , then  $C_k$  decomposes  $G[\overline{K_m}]$ .*

**Proof.** Since  $C_k$  decomposes  $G$ , by blowing up each cycle with  $\overline{K_m}$  we get a  $C_k[\overline{K_m}]$ -decomposition of  $G[\overline{K_m}]$ . Now by Theorem 1.4 we can decompose each of these  $k$ -partite graphs into  $k$ -cycles.  $\square$

Examining the decomposition of  $K_k[\overline{K_m}]$  into  $C_m$  one can easily observe that the existence for  $m$  even ( $m \geq 3$ ) follows immediately from Theorem 1.2, since each edge of  $K_k$  is blown up into a  $K_{m,m}$  and these can be decomposed into  $C_m$  for  $m$  even. For  $m$  odd and  $k$  even obviously no cycle decomposition is possible, since  $K_k[\overline{K_m}]$  is regular of odd degree. The remaining case for both  $m, k$  odd follows from a result by Ramírez-Alfonsín [10].

**Theorem 1.6.** *If  $m$  and  $k \geq 3$  are odd integers, then  $C_m$  decomposes  $C_k[\overline{K_m}]$ .*

Since for  $k$  odd  $K_k$  has a decomposition into Hamiltonian cycles  $C_k$  (Theorem 1.1), we can conclude the following theorem. The necessity is implied by the even-regularity condition.

**Theorem 1.7.** *Let  $m, k$  be integers,  $m \geq 3$ . The cycle  $C_m$  decomposes  $K_k[\overline{K_m}]$  if and only if either  $m$  is even or  $k$  is odd.*

Many other papers have been published on cycle decompositions of complete graphs  $K_n$  and of  $K_n - I$ , complete bi-, tri-, or  $n$ -partite graphs  $K_n[\overline{K_p}]$ ; see [4–6]. We mentioned just the results that are relevant to the constructions below. In this paper, we examine decompositions of complete graphs into graphs that arise by “blowing up” cycles. A natural extension of the results above is to look for a  $C_m[\overline{K_p}]$  decomposition of  $K_n$  and  $K_n - I$ . We restrict ourselves to  $p = 2$ . The graph  $C_m[\overline{K_2}]$  (denoted by  $C_m[2]$ ) which arises from the cycle  $C_m$  by replacing each vertex  $x$  by a pair of two independent vertices  $x', x''$  and each edge  $xy$  by four edges  $x'y', x'y'', x''y', x''y''$ . See Fig. 2.

Also a wide collection of papers has been published on decompositions of graphs into 2-factors, where a 2-factor is a vertex-disjoint and spanning collection of cycles of arbitrary lengths. For a recent survey, see [3]. An important tool in finding such decompositions is the following lemma by Häggkvist; see [8].

**Lemma 1.8.** *Let  $G$  be a path or a cycle with  $n$  edges and let  $H$  be a 2-regular graph on  $2n$  vertices with all components even. Then  $G[\overline{K_2}]$  can be decomposed into two edge-disjoint copies of  $H$ .*

In Section 1.2 we list necessary conditions for  $C_m[2]$  to decompose a complete graph. In Section 1.3 we give definitions of  $\rho$ - and  $\alpha$ -labelings, which are useful tools for decompositions of complete graphs. The following three sections solve the  $C_m[2]$  decomposition of  $K_{8mk+1}$  completely in three cases according to the value of  $k$ . In Section 2 we present direct constructions for any  $m$  and  $k = 1$ . In Section 3 we construct decompositions for  $m \equiv 1, 2, 3 \pmod{4}$  and  $k = 2$ . In Section 4 we first prove an auxiliary result for decompositions of complete multipartite graphs into  $C_m[2]$  and then use it for recursive constructions leading to decompositions of  $K_{8mk+1}$  into  $C_m[2]$  for any  $m \geq 3$  and  $k \geq 3$ . Finally, in Section 5.2 we show that a  $C_m[2]$  decomposition of  $K_{2n} - I$  is an easy corollary of Theorem 1.1. We will also show that our main result combined with Lemma 1.8 can yield new results in decompositions of complete graphs into non-uniform 2-regular subgraphs, i.e., unions of cycles of different lengths.

## 1.2. Necessary conditions

Necessary conditions for a  $C_m[2]$  to decompose  $K_n$  are

- (1)  $m \geq 3, n \geq 2m$ ,
- (2) the number of edges  $4m$  of  $C_m[2]$  divides the number of edges of  $K_n$ ,
- (3) the degree  $n - 1$  has to be a multiple of 4,
- (4)  $\frac{n-1}{2} \equiv 0 \pmod{4}$ .

Condition (4) follows from the fact that the number of edges  $\frac{n(n-1)}{2}$  of  $K_n$  has to be a multiple of 4 and  $n$  is odd. Thus for such a decomposition to exist we can conclude that

$$6 \leq 2m \leq n, \quad n \equiv 1 \pmod{8}, \quad \text{and} \quad m \mid \frac{n(n-1)}{8}.$$

We restrict ourselves only to the case when  $n = 8km + 1$  (notice that all the necessary conditions hold) and solve the restricted problem completely. The cases where  $6 \leq 2m \leq n$ ,  $n \equiv 1 \pmod{8}$ , and  $m \mid \frac{n(n-1)}{8}$  but  $m \nmid \frac{n-1}{8}$  remain open.

### 1.3. $\rho$ - and $\alpha$ -labelings

Let  $\lambda$  be an injection from the vertex set of a graph  $G$  with  $h$  edges into the set  $L = \{0, 1, \dots, 2h\}$  and let the length of every edge  $e = xy$  in  $G$  be determined by  $\ell(xy) = \min\{|\lambda(x) - \lambda(y)|, 2h + 1 - |\lambda(x) - \lambda(y)|\}$ . Then we say that the graph  $G$  has a  $\rho$ -labeling (sometimes called *rosy labeling*) if  $G$  contains edges of all lengths from 1 to  $h$ .

If  $G$  has a  $\rho$ -labeling such that only labels from  $L' = \{0, 1, \dots, h\}$  are used and there exists  $\lambda_0 \in \{0, 1, \dots, h\}$  such that for every edge  $xy \in E(G)$  with  $\lambda(x) < \lambda(y)$  it holds that  $\lambda(x) \leq \lambda_0 < \lambda(y)$ , then  $\lambda$  is called an  $\alpha$ -labeling. Obviously only bipartite graphs can admit  $\alpha$ -labelings.

It is well known (see [11]) that  $\rho$ - and  $\alpha$ -labelings yield decompositions of complete graphs as described in the following two theorems.

**Theorem 1.9.** *If a graph  $G$  with  $h$  edges has a  $\rho$ -labeling, then  $G$  decomposes  $K_{2h+1}$ .*

**Theorem 1.10.** *If a graph  $G$  with  $h$  edges has an  $\alpha$ -labeling, then  $G$  decomposes  $K_{2hk+1}$  for every positive integer  $k$ .*

While it is easy to use Theorem 1.10, some of the graphs we are interested in admit only  $\rho$ -labelings, but not  $\alpha$ -labelings. Therefore, we need to use  $\rho$ -labelings of other graphs than just a single copy of  $C_m[2]$ . We often will be using cyclic decompositions that will place  $k$  copies ( $k > 1$ ) of  $C_m[2]$  in the complete graph  $K_{8mk+1}$  at once. Notice, that we can allow distinct copies of  $C_m[2]$  to share vertices, but not edges. Let  $G$  be a graph that can be decomposed into  $k$  edge-disjoint copies of a graph  $H$  with  $h$  edges, say  $H_1, H_2, \dots, H_k$ . By Theorem 1.9,  $K_{2hk+1}$  can be decomposed into  $2hk + 1$  copies of  $G$ . Since each copy of  $G$  can be in turn decomposed into  $k$  edge-disjoint copies of  $H$ , it is obvious that  $K_{2hk+1}$  can be decomposed into  $(2hk + 1)k$  copies of  $H$ .

## 2. Decompositions of $K_{8mk+1}$ into $C_m[2]$ for $k = 1$

We are looking for decompositions of  $K_{8m+1}$  into  $C_m[2]$ . All constructions in Sections 2 and 3 are based on giving an  $\alpha$ - or  $\rho$ -labeling. In some cases it is enough to present a labeling of  $C_m$  and describe how to extend it into an  $\alpha$ -labeling or an  $\rho$ -labeling of  $C_m[2]$ . In some cases a more elaborate approach is required. If this is the case, we give a  $\rho$ -labeling of  $C_m[2]$ .

We distinguish four cases depending on the congruence class of  $m$  modulo 4.

**Lemma 2.1.** *If  $m \geq 3$  is an integer,  $m \equiv 0 \pmod{4}$ , then the graph  $C_m[2]$  admits an  $\alpha$ -labeling.*

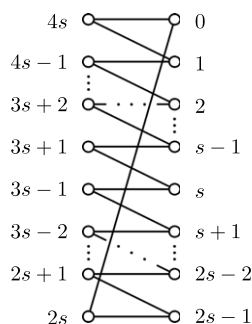
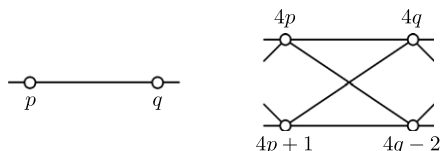
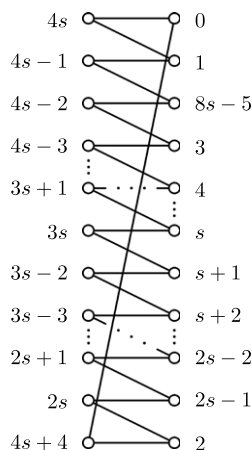
**Proof.** Let  $m \equiv 0 \pmod{4}$ , say  $m = 4s$ . An  $\alpha$ -labeling of  $C_m = C_{4s}$  is given in Fig. 1. It is easy to check that all lengths  $1, \dots, 4s$  are realized. The labeling is an  $\alpha$ -labeling where  $\lambda_0 = 2s - 1$ .

Then in  $C_{4s}$  we replace each vertex labeled  $p$ ,  $p \leq 2s - 1 = \lambda_0$ , by two vertices labeled  $4p$  and  $4p + 1$  and each vertex labeled  $q$ ,  $q > 2s - 1 = \lambda_0$ , by two vertices labeled  $4q$  and  $4q - 2$  (see Fig. 2). This way an edge  $(p, q)$  of length  $l = q - p$  will be replaced by four edges  $(4p + 1, 4q - 2)$ ,  $(4p, 4q - 2)$ ,  $(4p + 1, 4q)$ ,  $(4p, 4q)$  of lengths  $4l - 3 = 4q - 4p - 3$ ,  $4l - 2 = 4q - 4p - 2$ ,  $4l - 1 = 4q - 4p - 1$ , and  $4l = 4q - 4p$ . This clearly gives edges of all lengths  $1, 2, \dots, 16s = 4m$ . Hence,  $C_{4s}[2]$  has an  $\alpha$ -labeling with  $\lambda_0 = 8s - 3$ .  $\square$

**Lemma 2.2.** *If  $m \geq 3$  is an integer,  $m \equiv 2 \pmod{4}$ , then  $C_m[2]$  admits a  $\rho$ -labeling.*

**Proof.** Let  $m = 4s + 2$ . A  $\rho$ -labeling of  $C_m$  is given in Fig. 3.

The edges then have lengths  $4s, 4s - 1, 4s - 2, 4s - 3, 4s - 4, 4s - 5, 4s - 6, 4s - 7, \dots, 2s, 2s - 1, 2s - 3, 2s - 4, 2s - 5, 2s - 6, \dots, 3, 2, 1, 2s - 2, 4s + 2, 4s + 1$ . Then in  $C_{4s+2}$  again we replace each vertex labeled  $p$ ,  $p \leq 2s - 1$ , by two vertices labeled  $4p$  and  $4p + 1$ , the vertex labeled  $8s - 5$  by vertices labeled  $32s - 23, 32s - 22$  and each vertex labeled  $q$ ,  $2s - 1 < q \leq 4s + 4$ , by two vertices labeled  $4q$  and  $4q - 2$ . This way an edge  $(p, q)$  of length  $l = q - p$  (recall that  $p, q \neq 8s - 5$ ) will be replaced by four edges  $(4p + 1, 4q - 2)$ ,  $(4p, 4q - 2)$ ,  $(4p + 1, 4q)$ ,  $(4p, 4q)$  of lengths  $4l - 3 = 4q - 4p - 3$ ,  $4l - 2 = 4q - 4p - 2$ ,  $4l - 1 = 4q - 4p - 1$ ,  $4l = 4q - 4p$ . Thus, from edges of lengths  $l \in \{1, 2, \dots, 4s + 2\} \setminus \{4s - 3, 4s - 4\}$  we obtain all edges of lengths between 1 and  $16s + 8$  with the exception of lengths from  $16s - 19$  to  $16s - 12$ . But the missing lengths are covered by the edges  $(16s - 4, 32s - 23)$ ,  $(16s - 6, 32s - 23)$ ,  $(16s - 4, 32s - 22)$ ,  $(16s - 6, 32s - 22)$  and  $(16s - 8, 32s - 23)$ ,  $(16s - 10, 32s - 23)$ ,  $(16s - 8, 32s - 22)$ ,  $(16s - 10, 32s - 22)$  that arise from the edges  $(4s - 1, 8s - 5)$  and  $(4s - 2, 8s - 5)$ . This clearly gives edges of all lengths  $1, 2, \dots, 16s + 8 = 4(4s + 2) = 4m$ .  $\square$

Fig. 1.  $\alpha$ -labeling of  $C_{4s}$ .Fig. 2. Blowing up an edge  $(p, q)$ .Fig. 3.  $\rho$ -labeling of  $C_{4s+2}$ .

**Lemma 2.3.** If  $m \geq 3$  is an integer,  $m \equiv 1 \pmod{4}$ , then  $C_m[2]$  admits a  $\rho$ -labeling.

**Proof.** Let  $m = 4s + 1$ . For small values  $s = 1, 2, \dots, 7$  the  $\rho$ -labeling is given in Figs. 4 and 5. In Fig. 6 we give a general construction for any  $s \geq 8$ . It is easy to verify that all lengths  $1, 2, \dots, 16s + 4 = 4m$  are realized and thus it is a  $\rho$ -labeling.  $\square$

**Lemma 2.4.** Is  $m \geq 3$  is an integer,  $m \equiv 3 \pmod{4}$ , then  $C_m[2]$  admits a  $\rho$ -labeling.

**Proof.** Let  $m = 4s + 3$ . The  $\rho$ -labeling of  $C_m$  is given in Fig. 7. The edges have lengths  $4s + 1, 4s, 4s - 1, 4s - 2, 4s - 3, \dots, 2s + 3, 2s + 2, 2s, 2s - 1, 2s - 2, 2s - 3, \dots, 4, 3, 2, 2s + 1, 4s + 3, 1, 4s + 2$ .

Then in  $C_{4s+3}$  we replace each vertex labeled  $p, p \leq 2s - 1$ , by two vertices labeled  $p$  and  $p + m$ ; the vertex labeled  $6s + 4$  by vertices labeled  $18s + 13$  and  $22s + 16$ ; each vertex labeled  $q, 2s - 1 < q \leq 4s + 1$  and  $q = 4s + 3$ , by two vertices labeled  $m + q$  and  $3m + q$ , respectively; and the vertex labeled  $4s + 2$  by vertices labeled  $4m - 1$  and  $3m + 1$ . This way an edge  $(p, q)$  of length  $l = q - p$  (recall that  $p, q \neq 4s + 2, 6s + 4$ ) will be replaced by four edges  $(m + q, p + m), (m + q, p), (3m + q, p + m), (3m + q, p)$  of lengths  $q - p = l, q - p + m = l + m, q - p + 2m = l + 2m, q - p + 3m = l + 3m$ .

Therefore the edges of lengths  $l \in \{1, 2, \dots, 4s + 2, 4s + 3\} \setminus \{4s + 3, 4s + 2, 2s + 1, 1\}$  after blowing up give edges of all lengths between 1 and  $16s + 12$  with the exception of lengths  $4s + 3, 4s + 3 + m = 8s + 6, 4s + 3 + 2m = 12s + 9, 4s + 3 + 3m = 16s + 12, 4s + 2, 4s + 2 + m = 8s + 5, 4s + 2 + 2m = 12s + 8, 4s + 2 + 3m = 16s + 11, 2s + 1, 2s + 1 + m = 6s + 4, 2s + 1 + 2m = 10s + 7, 2s + 1 + 3m = 14s + 10, 1, 1 + m = 4s + 4, 1 + 2m = 8s + 7, 1 + 3m = 12s + 10$ .

The remaining edges of lengths  $4s + 2, 1$  and  $4s + 3, 2s + 1$ , respectively, which are incident with the vertices  $4s + 2$  and  $6s + 4$  in  $C_{4s+3}$ , will be blown up as follows. The edges  $(2m, 4m - 1), (4m, 4m - 1), (2m, 3m + 1), (4m, 3m + 1)$

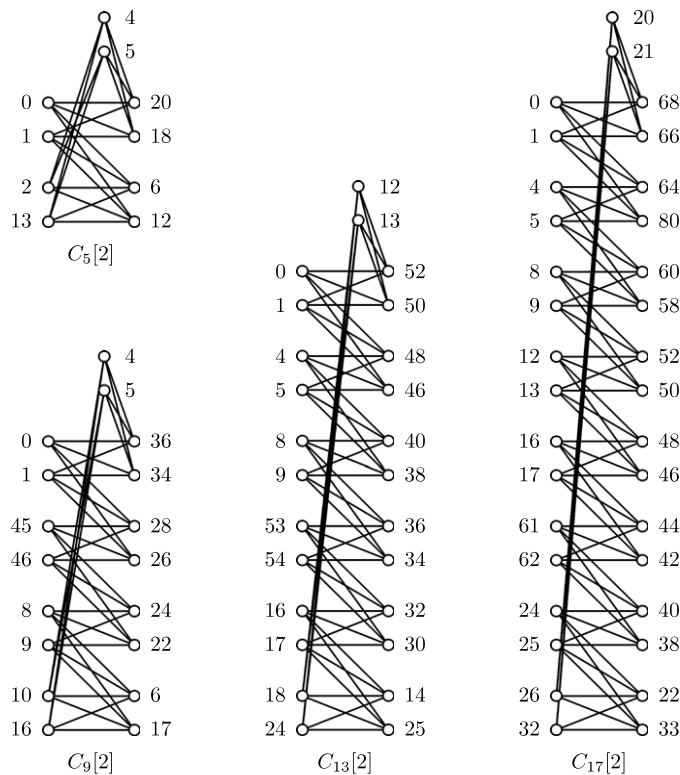


Fig. 4.  $\rho$ -labeling of  $C_{4s+1}[2]$  for  $s = 1, 2, 3, 4$ .

arising from the edge  $(4s + 3, 4s + 2)$  have lengths  $2m - 1 = 8s + 5$ ,  $1, m + 1 = 4s + 4$ ,  $m - 1 = 4s + 2$ . The edges  $(4m - 1, 0)$ ,  $(4m - 1, m)$ ,  $(3m + 1, 0)$ ,  $(3m + 1, m)$  arising from the edge  $(4s + 2, 0)$  have lengths  $4m - 1 = 16s + 11$ ,  $3m - 1 = 12s + 8$ ,  $3m + 1 = 12s + 10$ ,  $2m + 1 = 8s + 7$ . The edges  $(2s + 1 + m, 18s + 13)$ ,  $(2s + 1 + 3m, 18s + 13)$ ,  $(2s + 1 + m, 22s + 16)$ ,  $(2s + 1 + 3m, 22s + 16)$  arising from  $(2s + 1, 6s + 4)$  have lengths  $16s + 12 - m = 12s + 9$ ,  $16s + 12 - 3m = 4s + 3$ ,  $20s + 15 - m = 16s + 12$ ,  $20s + 15 - 3m = 8s + 6$ . Finally, the edges  $(4s + 3 + m, 18s + 13)$ ,  $(4s + 3 + 3m, 18s + 13)$ ,  $(4s + 3 + m, 22s + 16)$ ,  $(4s + 3 + 3m, 22s + 16)$  arising from  $(4s + 3, 6s + 4)$  have lengths  $14s + 10 - m = 10s + 7$ ,  $14s + 10 - 3m = 2s + 1$ ,  $18s + 13 - m = 14s + 10$ ,  $18s + 13 - 3m = 6s + 4$ .

Hence, from above it follows that  $C_{4s+3}[2]$  has a  $\rho$ -labeling and the proof is complete.  $\square$

The next theorem follows immediately from Lemmas 2.1–2.4 and Theorems 1.9 and 1.10.

**Theorem 2.5.** The graph  $C_m[2]$  decomposes  $K_{8m+1}$  for every  $m \geq 3$ .

We proved in Lemma 2.1 that  $C_m[2]$  for  $m \equiv 0 \pmod{4}$  admits an  $\alpha$ -labeling. Therefore, the following theorem is an immediate consequence of Lemma 2.1 and Theorem 1.10. It gives a complete characterization of decompositions of  $K_{8km+1}$  into  $C_m[2]$  for the case of  $m \equiv 0 \pmod{4}$ .

**Theorem 2.6.** The graph  $C_m[2]$  with  $m \equiv 0 \pmod{4}$  decomposes  $K_{8mk+1}$  for every  $k > 0$  and  $m \geq 3$ .

Thus we can omit this case in the following sections.

### 3. Decompositions of $K_{8mk+1}$ into $C_m[2]$ for $k = 2$

For a  $C_m[2]$  decomposition of  $K_{8km+1}$  when  $k = 2$  we use different techniques based on the congruence class of  $m$  modulo 4. For  $m \equiv 0 \pmod{4}$  the decomposition follows from Theorem 2.6. For  $m \equiv 2 \pmod{4}$  we give a labeling for  $C_m$  and show how to extend it to a  $\rho$ -labeling of edge-disjoint union of  $k$  copies of  $C_m[2]$ . Finally for  $m \equiv 1, 3 \pmod{4}$  we give  $\rho$ -labelings for an edge-disjoint union of 2 copies of  $C_m[2]$  directly.

**Lemma 3.1.** If  $k \geq 2$  and  $m \geq 3$  are integers,  $m \equiv 2 \pmod{4}$ , then there exists an edge-disjoint union of  $k$  copies of  $C_m[2]$  that admits a  $\rho$ -labeling.

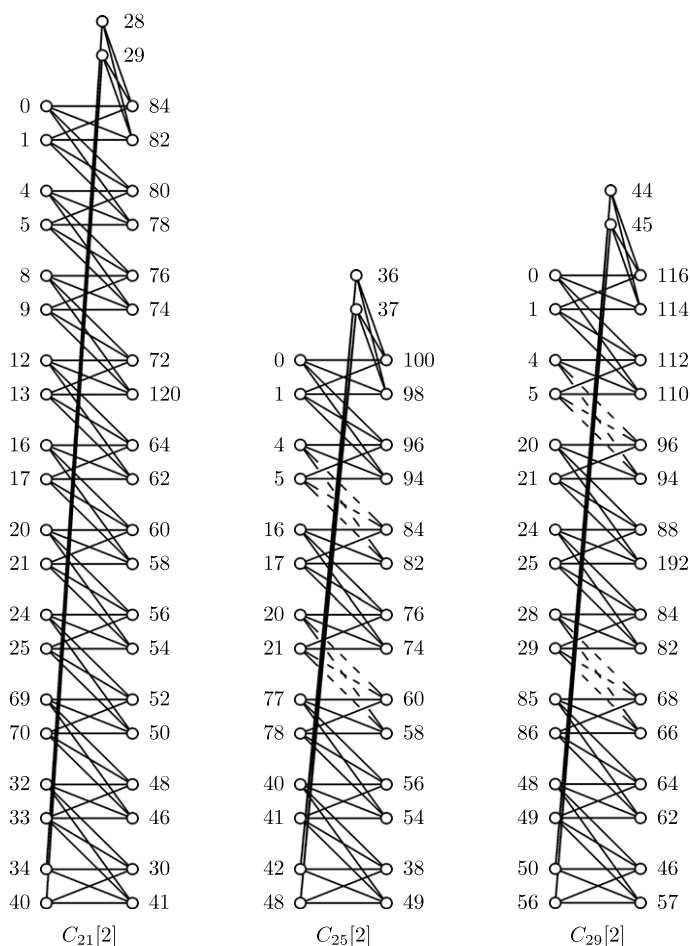


Fig. 5.  $\rho$ -labeling of  $C_{4s+1}[2]$  for  $s = 5, 6, 7$ .

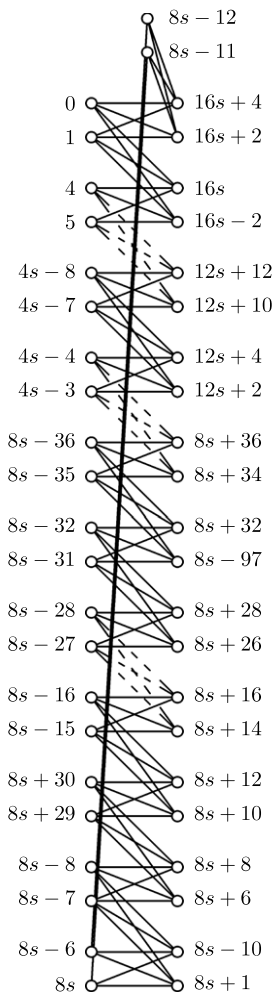
**Proof.** Let  $m = 4s + 2$ . We want to show that the edge-disjoint union of  $k$  copies of  $C_{4s+2}$  admits a  $\rho$ -labeling for  $k > 1$  and  $s \geq 1$ . Thus, we give a labeling of the edge-disjoint union of  $k$  copies of  $C_{4s+2}$  that contains edges of all lengths from 1 to  $4ks + 2k$ .

First let  $C_m^i = C_{4s+2}^i$  for  $i = 0, 1, \dots, k-1$  be labeled as in Fig. 9, where  $C_{4s+2}^i$  stands for  $i$ th copy of  $C_{4s+2}$  in edge-disjoint union of  $k$  copies of  $C_{4s+2}$ . This construction in Fig. 9 yields a  $\rho$ -labeling only for  $s > 1$ , the case  $s = 1$  is treated separately.

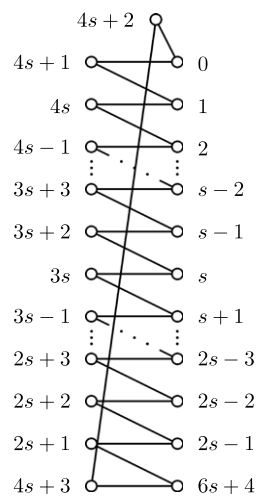
The path  $P = 4is + 4s - 2, 4i + 3, 4is + 4s - 3, 4i + 4, \dots, 4is + 3s, 4i + s + 1, 4is + 3s - 2, 4i + s + 2, 4is + 3s - 3, 4i + s + 3, \dots, 4is + 2s + 2, 4i + 2s - 2, 4is + 2s + 1, 4i + 2s - 1, 4is + 2s, 4i + 2$  contains the edges of lengths  $4is + 4s - 4i - 5, 4is + 4s - 4i - 6, 4is + 4s - 4i - 7, \dots, 4is + 2s - 4i - 1, 4is + 2s - 4i - 3, 4is + 2s - 4i - 4, 4is + 2s - 4i - 5, 4is + 2s - 4i - 6, \dots, 4is - 4i + 4, 4is - 4i + 3, 4is - 4i + 2, 4is - 4i + 1, 4is + 2s - 4i - 2$ . We see that the edges listed above cover all lengths between 1 and  $4ks - 4k$  with the exception of lengths  $4s - 4, 8s - 8, 12s - 12, \dots, 4ks - 4k$ . But the missing lengths are covered by the edges  $(4ks - 3k + 3i + 2, 4ks + 4is + 4s - 3k - i - 2)$  of lengths  $4is + 4s - 4i - 4$ . Thus, we have edges of all lengths from 1 to  $4ks - 4k$ .

The edges  $(4ks - 3k + 3i + 2, 2i + 1)$  and  $(4is + 4s - 2, 4ks + 4is + 4s - 3k - i - 2)$  of lengths  $4ks - 3k + i + 1$  and  $4ks - 3k - i$  cover all lengths between  $4ks - 4k + 1$  and  $4ks - 2k$ . The edges  $(4ks - k + i + 1, 0)$  and  $(4ks - k + i + 1, 2i + 1)$  of lengths  $4ks - k + i + 1$  and  $4ks - k - i$  cover all lengths between  $4ks - 2k + 1$  and  $4ks$ . Finally, the edges  $(4ks + 2k + 2i + 2, 0 = 8ks + 4k + 1)$  and  $(4ks + 2k + 2i + 2, 4i + 2)$  of lengths  $4ks + 2k - 2i - 1$  and  $4ks + 2k - 2i$  cover all lengths from  $4ks + 1$  to  $4ks + 2k$ . It follows that the edge-disjoint union of  $k$  copies of  $C_{4s+2}$  has a  $\rho$ -labeling for  $k, s > 1$ .

Similarly as in the proof of Lemma 2.2 we replace each vertex labeled  $p, p \leq 4i + 2s$ , by two vertices labeled  $4p$  and  $4p + 1$ ; the vertex labeled  $4ks + 4is + 4s - 3k - i - 2$  by vertices labeled  $16ks + 16is + 16s - 12k - 4i - 10, 16ks + 16is + 16s - 12k - 4i - 11$ ; and each of the remaining vertices labeled  $q, 4i + 2s - 1 < q$ , by two vertices labeled  $4q$  and  $4q - 2$ . Hence, every edge  $(p, q)$  of length  $l = q - p$  for  $p, q \neq 4ks + 4is + 4s - 3k - i - 2$  will be replaced by four edges of lengths  $4l, 4l - 1, 4l - 2, 4l - 3$ . We still have to check that two edges of lengths  $4is + 4s - 4i - 4$  and  $4ks - 3k - i$ , which are incident with the vertex  $4ks + 4is + 4s - 3k - i - 2$  give, after blowing up their endvertices, the edges of lengths  $16is + 16s - 16i - 16 - r$  and  $16ks - 12k - 4i - r$ , where  $r = 0, 1, 2, 3$ .



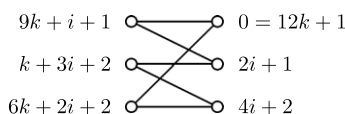
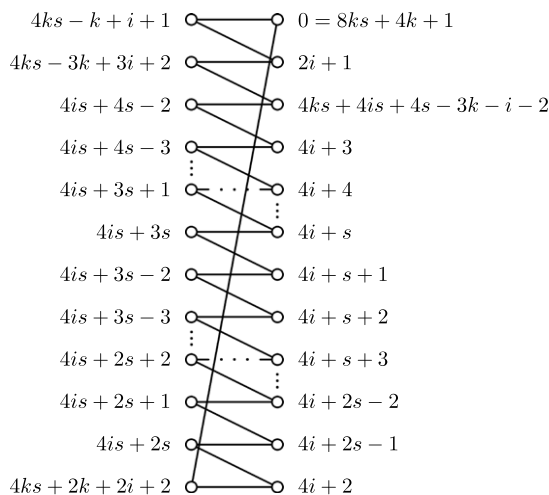
**Fig. 6.**  $\rho$ -labeling of  $C_m[2]$  for  $m = 4s + 1$  and  $s \geq 8$ .



**Fig. 7.**  $\rho$ -labeling of  $C_{4s+3}$ .

The edge  $(4ks - 3k + 3i + 2, 4ks + 4is + 4s - 3k - i - 2)$  is replaced by edges  $(16ks - 12k + 12i + 6, 16ks + 16is + 16s - 12k - 4i - 10)$ ,  $(16ks - 12k + 12i + 6, 16ks + 16is + 16s - 12k - 4i - 11)$ ,  $(16ks - 12k + 12i +$



Fig. 8. Labeling of  $C_6^i$ .Fig. 9. Labeling of  $C_{4s+2}^i$ .

8,  $16ks + 16is + 16s - 12k - 4i - 10$ ,  $(16ks - 12k + 12i + 8, 16ks + 16is + 16s - 12k - 4i - 11)$  of lengths  $16is + 16s - 16i - 16$ ,  $(16is + 16s - 16i - 16) - 1$ ,  $(16is + 16s - 16i - 16) - 2$ ,  $(16is + 16s - 16i - 16) - 3$ , respectively.

The edge  $(4is + 4s - 2, 4ks + 4is + 4s - 3k - i - 2)$  is replaced by edges  $(16is + 16s - 10, 16ks + 16is + 16s - 12k - 4i - 10)$ ,  $(16is + 16s - 10, 16ks + 16is + 16s - 12k - 4i - 11)$ ,  $(16is + 16s - 8, 16ks + 16is + 16s - 12k - 4i - 10)$ ,  $(16is + 16s - 8, 16ks + 16is + 16s - 12k - 4i - 11)$  of lengths  $16ks - 12k - 4i$ ,  $(16ks - 12k - 4i) - 1$ ,  $(16ks - 12k - 4i) - 2$ ,  $(16ks - 12k - 4i) - 3$ . Thus the edge-disjoint union of  $k$  copies of  $C_{4s+2}[2]$  has edges of all lengths from 1 to  $16ks + 8k = 4k(4s + 2)$  and therefore the edge-disjoint union of  $k$  copies of  $C_{4s+2}[2]$  admits a  $\rho$ -labeling for  $k, s > 1$ .

Now we have to show that the edge-disjoint union of  $k$  copies of  $C_{4s+2}$  admits a  $\rho$ -labeling also for  $s = 1$  and  $k > 1$ . Let every cycle  $C_6^i$ ,  $i = 0, 1, \dots, k - 1$ , of an edge-disjoint union of  $k$  copies of  $C_6$  be labeled so that  $C_6^i = 0, 9k + i + 1, 2i + 1, k + 3i + 2, 4i + 2, 6k + 2i + 2, 0$  (see Fig. 8). Then the edges  $(k + 3i + 2, 4i + 2)$  and  $(k + 3i + 2, 2i + 1)$  of lengths  $k - i$  and  $k + i + 1$  cover all lengths between 1 and  $2k$ , the edges  $(9k + i + 1, 2i + 1)$  and  $(9k + i + 1, 0)$  of lengths  $3k + i + 1$  and  $3k - i$  cover all lengths from  $2k + 1$  to  $4k$ , and the edges  $(6k + 2i + 2, 4i + 2)$  and  $(6k + 2i + 2, 0 = 12k + 1)$  of lengths  $6k - 2i$  and  $6k - 2i - 1$  cover all lengths between  $4k + 1$  and  $6k$ .

If we replace in the edge-disjoint union of  $k$  copies of  $C_6$  each vertex labeled  $p$ ,  $p = 0, 2i + 1, 4i + 2$ , by two vertices  $4p$  and  $4p + 1$  and each vertex labeled  $q$ ,  $q = k + 3i + 2, 6k + 2i + 2, 9k + i + 1$ , by two vertices  $4q$  and  $4q - 2$ , then we obtain an edge-disjoint union of  $k$  copies of  $C_6[2]$  with a  $\rho$ -labeling.  $\square$

We illustrate the lemma for  $k = 2$  in Fig. 10. Note that while we have proved an existence of a  $\rho$ -labeling of an edge-disjoint union of  $k$  copies of  $C_m[2]$ , in the figure we are showing the labeling of the two edge-disjoint copies of  $C_m[2]$  separately. Therefore, some labels repeat in both copies of  $C_m[2]$ . We will use this approach for the reader's convenience in our figures throughout this section.

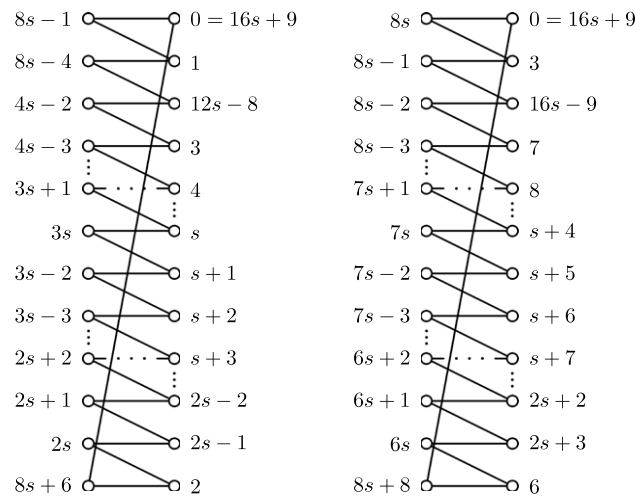
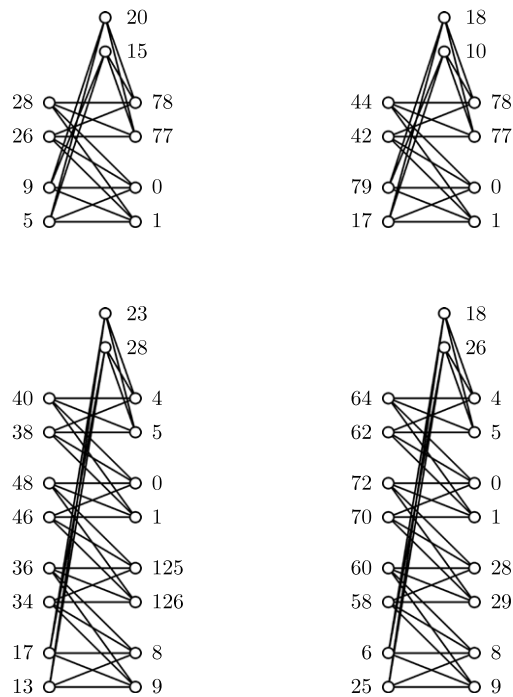
**Lemma 3.2.** *If  $m \geq 3$  is an integer,  $m \equiv 1 \pmod{4}$ , then there exists an edge-disjoint union of two copies of  $C_m[2]$  that admits a  $\rho$ -labeling.*

**Proof.** Let  $m = 4s + 1$ . For small  $s$  the decomposition is given by a  $\rho$ -labeling (see Fig. 11). In Fig. 12 we give a general construction of a  $\rho$ -labeling for any  $s \geq 3$ . Vertex labels for a fixed  $s$  will repeat only for (most) vertices placed on the right. There is no edge between any two of them and one can check that no edge of the complete graph is covered more than once by evaluating the edge lengths and verifying that all lengths  $1, 2, \dots, 32s + 8$  are realized.  $\square$

**Lemma 3.3.** *If  $m \geq 3$  is an integer,  $m \equiv 3 \pmod{4}$ , then there exists an edge-disjoint union of two copies of  $C_m[2]$  that admits a  $\rho$ -labeling.*

**Proof.** Let  $m = 4s + 3$ . For small  $s$  the decomposition is given by a  $\rho$ -labeling (see Fig. 13). In Fig. 14 we give a general construction of a  $\rho$ -labeling for any  $s \geq 3$ . Again it can be checked in the figures that all lengths  $1, 2, \dots, 32s + 24$  are realized.  $\square$



Fig. 10.  $\rho$ -labeling of a union of a pair of  $C_{4s+2}$ .Fig. 11.  $\rho$ -labeling of a union a pair of  $C_5[2]$  and a pair of  $C_9[2]$ .

From Lemmas 3.1–3.3, Theorems 2.6, 1.9 and 1.10 we have the following theorem.

**Theorem 3.4.** The graph  $C_m[2]$  decomposes  $K_{8mk+1}$  for  $k = 2$  and for every  $m \geq 3$ .

#### 4. Decompositions of $K_{8mk+1}$ into $C_m[2]$ for $k \geq 3$

In this section we use a different approach to construct the  $C_m[2]$  decomposition of  $K_{8mk+1}$  for any  $m \geq 3$  and any  $k \geq 3$ . Since the following construction does not depend on the modularity of  $m$ , some of the results below will cover cases discussed already in Sections 2 and 3, namely the  $C_m[2]$  decomposition of  $K_{8mk+1}$  for  $m \equiv 0, 2 \pmod{4}$ . Nonetheless we give the statements in their general form rather than just rephrasing them for the missing cases only.

We do the decomposition in two stages. First we decompose the graph  $K_{8mk+1}$  into  $k$  copies of  $K_{8m+1}$  (all sharing exactly one vertex) and one complete multipartite graph with  $k$  partite sets and  $8m$  vertices in each partite set. Then we decompose

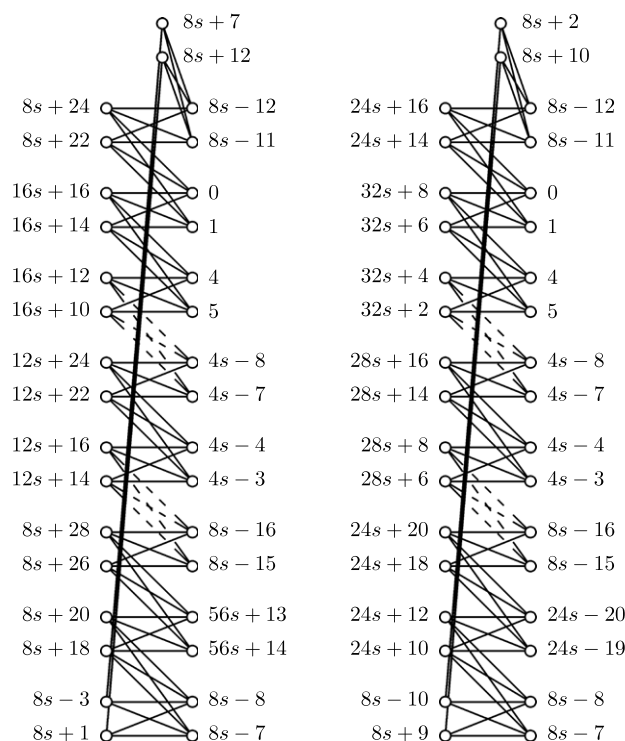


Fig. 12.  $\rho$ -labeling of a union of a pair of  $C_{4s+1}[2]$ ,  $s \geq 3$ .

each of the graphs  $K_{8m+1}$  into  $C_m[2]$  using the constructions from Section 2. We are left with the problem of decomposing the complete multipartite graph  $K_k[\overline{K}_{8m}]$  into  $C_m[2]$ . See Fig. 15.

We can state this observation as the following lemma.

**Lemma 4.1.** *Let  $m, k$  be integers,  $m \geq 3$ . If  $C_m[2]$  decomposes  $K_k[\overline{K}_{8m}]$ , then  $C_m[2]$  decomposes  $K_{8mk+1}$ .*

**Proof.** In Section 2 it was proved that  $C_m[2]$  decomposes  $K_{8m+1}$  for every  $m \geq 3$ . Therefore a decomposition of  $K_k[\overline{K}_{8m}]$  into  $C_m[2]$  together with a  $C_m[2]$  decomposition of each of the  $k$  copies of  $K_{8m+1}$  gives a decomposition of  $K_{8mk+1}$  into  $C_m[2]$ .  $\square$

Now combining the results from Section 1 we show the following

**Lemma 4.2.** *If  $m$  and  $k$  are integers,  $m, k \geq 3$ , then  $C_m[2]$  decomposes the multipartite graph  $K_k[\overline{K}_{8m}]$ .*

**Proof.** We notice that  $K_k[\overline{K}_2] \cong K_{2k} - I$ . The number of edges of  $K_{2k} - I$  is  $2k(k-1)$ ; therefore, by Theorem 1.1  $K_k[\overline{K}_2]$  can be decomposed into both  $C_k$  and  $C_{k-1}$ . Obviously one of the numbers  $k$  and  $k-1$  is odd. We distinguish two cases.

- (1) If  $k$  is odd ( $k \geq 3$ ) then we can decompose  $K_k[\overline{K}_2]$  into  $C_k$  and we blow up each  $C_k$  by  $m$ . Since  $k$  is odd, each of the  $C_k[m]$  can be decomposed into  $C_m$  by Theorem 1.6 for  $m$  odd and by Corollary 1.3 for  $m$  even. Hence, we have  $C_k[m]$ -decomposition of  $K_k[\overline{K}_{2m}]$ . Now by blowing up by  $\overline{K}_2$  and by Corollary 1.5 we can have a  $C_m$  decomposition of  $K_k[\overline{K}_{4m}]$ . Finally by blowing up again by  $\overline{K}_2$  we get a  $C_m[2]$  decomposition of  $K_k[\overline{K}_{8m}]$ .
- (2) If  $k$  is even then  $k-1$  is odd and for  $k > 2$  the proof goes similarly. We decompose  $K_k[\overline{K}_2]$  into  $C_{k-1}$ . Each of the  $C_{k-1}$  can be blown up by  $m$  and by Theorem 1.6 for  $m$  odd and by Corollary 1.3 for  $m$  even we get a  $C_m$  decomposition of  $K_k[\overline{K}_{2m}]$ . Again by blowing up by  $\overline{K}_2$  and by Corollary 1.5 we can have a  $C_m$  decomposition of  $K_k[\overline{K}_{4m}]$  and blowing up again by  $\overline{K}_2$  we get a  $C_m[2]$  decomposition of  $K_k[\overline{K}_{8m}]$ .  $\square$

Finally from the previous two lemmas immediately follows

**Theorem 4.3.** *If  $m \geq 3$  is an integer, then  $C_m[2]$  decomposes  $K_{8mk+1}$  for any integer  $k \geq 3$ .*

## 5. Conclusion

### 5.1. Summary

Combining the results from Sections 2–4 we conclude

**Theorem 5.1.** *If  $m \geq 3$  is an integer, then  $C_m[2]$  decomposes  $K_{8mk+1}$  for every integer  $k > 0$ .*

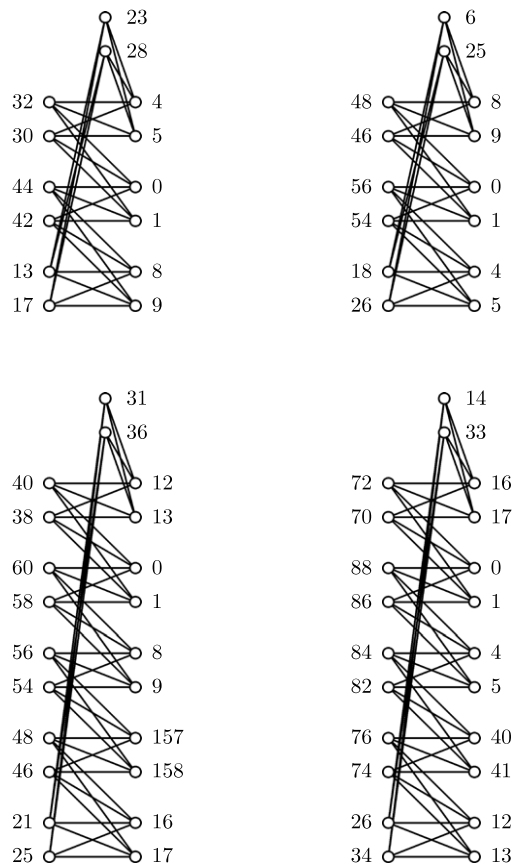


Fig. 13.  $\rho$ -labeling of a union of a pair of  $C_7[2]$  and a pair of  $C_{11}[2]$ .

**Proof.** The claim follows for  $m \equiv 0 \pmod{4}$  from Theorem 2.6 and for  $m \equiv 1, 2, 3 \pmod{4}$  from Theorems 2.5, 3.4 and 4.3.  $\square$

### 5.2. Decomposition of $K_{2n} - I$

Analogously to the results mentioned in Section 1, one can ask about  $C_m[2]$  decompositions of  $K_n - I$ . Clearly, if we want to decompose a complete graph  $K_n - I$  into  $C_m[2]$ , the degree  $n - 2$  of every vertex has to be a multiple of four. The necessary conditions are  $2m \leq n$ ,  $4 \mid (n - 2)$  and  $2m \mid \frac{n(n-2)}{4}$ . An easy observation shows that the necessary conditions are also sufficient.

**Theorem 5.2.** Let  $m \geq 3$  be an integer. The graph  $C_m[2]$  decomposes  $K_n - I$  if and only if  $n$  is an integer such that  $n \geq 2m$ ,  $n \equiv 2 \pmod{4}$  and  $2m \mid \frac{n(n-2)}{4}$ .

**Proof.** The necessity was shown above. The sufficiency follows from Theorem 1.1. We take  $p = \frac{n}{2} = \frac{4t+2}{2} = 2t + 1$  for some  $t > 0$ . Obviously  $p$  is odd and from  $2m \mid \frac{n(n-2)}{4}$  follows  $2m \mid p(p - 1)$ . Now from Theorem 1.1 it follows that  $C_m$  decomposes  $K_p$  whenever  $p$  is odd and  $2m \mid p(p - 1)$ . By blowing up the graph  $K_p$  as well as each cycle  $C_m$  by  $\overline{K}_2$ , we get a  $C_m[2]$  decomposition of  $K_p[\overline{K}_2] \cong K_{2p} - I = K_n - I$  for all feasible values of  $n$ .  $\square$

### 5.3. Non-uniform cycle decomposition

Taking  $p = 2m$  in Theorem 5.1 (or Theorem 5.2, respectively) and combining it with Lemma 1.8, decompositions of a complete graph  $K_n$  (or  $K_n - I$ , respectively) into certain collections of even cycles can be obtained.

**Corollary 5.3.** Let  $m \geq 3$  be an integer. If  $C_{p_1}, C_{p_2}, \dots, C_{p_t}$  is a collection of even cycles such that  $\sum_{i=1}^t p_i = 2m$ , then the complete graph  $K_{8m+1}$  can be decomposed into a vertex-disjoint union of cycles  $C_{p_1}, C_{p_2}, \dots, C_{p_t}$ .

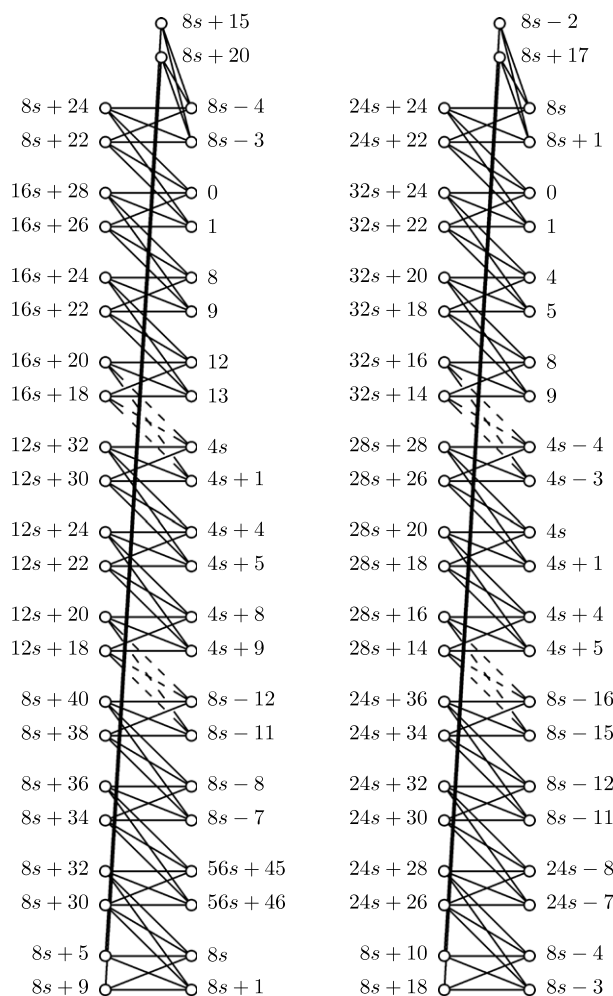


Fig. 14.  $\rho$ -labeling of a union of a pair of  $C_{4s+3}[2]$ ,  $s \geq 3$ .

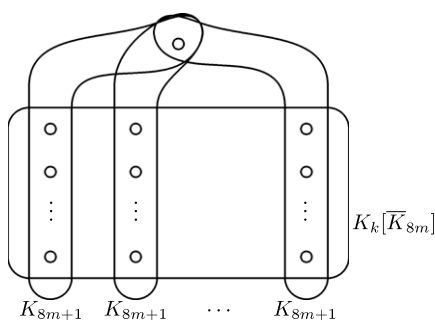


Fig. 15. Decomposition of  $K_{8mk+1}$  into an edge-disjoint union of  $k$  copies of  $K_{8m+1}$  and  $K_k[\overline{K}_{8m}]$ .

**Corollary 5.4.** If  $C_{p_1}, C_{p_2}, \dots, C_{p_t}$  is a collection of even cycles such that  $\sum_{i=1}^t p_i = p \leq 4n + 2$  and  $p \mid n(4n + 2)$ , then the graph  $K_{4n+2} - I$  can be decomposed into a vertex-disjoint union of cycles  $C_{p_1}, C_{p_2}, \dots, C_{p_t}$ .

While Corollary 5.4 is an easy application of Haggkvist lemma (Lemma 1.8) to blown up complete graphs  $K_{2p} - I \simeq K_p[2]$ , Corollary 5.3 extends the same technique also to certain complete graphs. In particular we observe that the complete graph  $K_{8m+1}$  can be decomposed into any collection of even cycles whose lengths sum up to  $p$ . This was not covered by any known method; see also [7]. Notice that without loss of generality we could omit the parameter  $k$  since we decompose the graph  $C_m[2]$  into any smaller even cycles.

An interesting observation is that each of the  $8m + 1$  graphs  $C_m[2]$  in  $K_{8m+1}$  can be decomposed into a different collection of even cycles, each in an even number of copies. Hence the following corollary. By  $2C_{p_{i,j}}$  we denote a pair of edge-disjoint cycles of length  $p_{i,j}$ .

**Corollary 5.5.** *Let  $m \geq 3$  be an integer. For every  $i = 1, 2, \dots, 8m + 1$  let  $t_i$  and  $p_{i,j}$ ,  $j = 1, 2, \dots, t_i$  be numbers such that every  $p_{i,j}$  is even and  $\sum_{j=1}^{t_i} p_{i,j} = 2m$ . Then  $D = \{2C_{p_{i,j}} : i = 1, 2, \dots, 8m + 1, j = 1, 2, \dots, t_i\}$  is a collection of cycles which form a decomposition of the complete graph  $K_{8m+1}$ .*

Alspach conjectured, see [7], that for  $n$  odd every  $K_n$  can be decomposed into any collection of cycles as long as the cycle lengths do not exceed  $n$  and their lengths sum up to  $n(n-1)/2$ . Corollary 5.5 brings a new set of results not known before. Unfortunately, only values  $n \equiv 1 \pmod{8}$  are addressed, since for  $n \equiv 3, 5, 7 \pmod{8}$  necessary conditions for  $C_m[2]$  to decompose the complete graph are not satisfied.

#### 5.4. Open problems

The problem presented in this article is a natural extension of recent results (see Theorem 1.1) by Alspach and Gavlas [1] and Šajna [12]. In Section 1.2 necessary conditions for  $C_m[2]$  to decompose  $K_n$  were given as

$$6 \leq 2m \leq n, \quad n \equiv 1 \pmod{8}, \quad \text{and} \quad m \mid \frac{n(n-1)}{8}.$$

We solved the problem for  $n = 8km + 1$  completely.

The missing case (see 1.2) where  $6 \leq 2m \leq n$ ,  $n \equiv 1 \pmod{8}$ , and  $m \mid \frac{n(n-1)}{8}$  while  $m \nmid \frac{n-1}{8}$  remains open. We believe that such a decomposition is possible, though we did not attempt to find any general construction. We mention only that the smallest case when  $C_3[2]$  should decompose  $K_9$  is not possible.

Another possible extension of the presented problem is to look for  $C_m[r]$  decompositions of  $K_n$ .

#### Acknowledgements

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#### References

- [1] B. Alspach, H. Gavlas, Cycle decompositions of  $K_n$  and  $K_n - I$ , J. Combin. Theory, Ser. B 81 (2001) 77–99.
- [2] J.C. Bermond, C. Huang, D. Sotteau, Balanced cycle and circuit designs: Even case, Ars. Combin. 5 (1978) 293–318.
- [3] D. Bryant, Cycle decompositions of complete graphs, in: Surveys in Combinatorics 2007, in: London Math. Soc. Lecture Note Ser., vol. 346, Cambridge Univ Press, Cambridge, 2007, pp. 67–97.
- [4] N. Cavenagh, Decompositions of complete tripartite graphs into  $k$ -cycles, Australas. J. Combin. 18 (1998) 193–200.
- [5] N.J. Cavenagh, E.J. Billington, Decompositions of complete multipartite graphs into cycles of even length, Graphs Combin. 16 (2000) 49–65.
- [6] J.E. Cockayne, B.L. Hartnell, Edge partitions of complete multipartite graphs into equal length circuits, J. Combin. Theory, Ser. B 23 (1977) 174–183.
- [7] C.J. Colbourn, J.H. Dinitz (Eds.), The CRC Handbook of Combinatorial Designs, 2nd ed., Discrete Mathematics and its Applications, Boca Raton, FL, 2007.
- [8] R. Häggkvist, A lemma on cycle decompositions, North-Holland Math. Study 115 (1985) 227–232.
- [9] A. Muthusamy, P. Paulraja, Factorizations of product graphs into cycles of uniform length, Graphs Combin. 11 (1995) 69–90.
- [10] J.L. Ramírez-Alfonsín, Cycle decompositions of complete and complete multipartite graphs, Australas. J. Combin. 11 (1995) 233–238.
- [11] A. Rosa, On certain valuations of the vertices of a graph, in: Theory Graphs, Int. Symp. Rome 1966, Gordon and Breach, NY and Dunod Paris, 1967, pp. 349–355.
- [12] M. Šajna, Cycle decompositions III, J. Combin. Des. 10 (2002) 27–78.
- [13] D. Sotteau, Decompositions of  $K_{m,n}$ ,  $K_{m,n}^*$  into cycles (circuits) of length  $2k$ , J. Combin. Theory, Ser. B 30 (1981) 75–81.